Physics 606 Exam 1

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your work.

An answer, even if correct, will receive zero credit unless it is obtained via the work shown.

Do all your work on the blank sheets provided, writing your name clearly, and turn them in stapled together. You may keep these questions.

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,x + \frac{i}{\sqrt{2m\hbar\omega}}\,p \quad , \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,x - \frac{i}{\sqrt{2m\hbar\omega}}\,p \quad , \quad N = a^\dagger a$$

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \quad , \quad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad , \quad \vec{p} \to -i\hbar\vec{\nabla}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad [\sigma_i, \sigma_j] = 2\,i\,\varepsilon_{ijk}\,\sigma_k$$

$$\left[\hat{p}_i, F(\hat{\mathbf{r}})\right] = -i\hbar\frac{\partial F}{\partial \hat{x}_i} \quad \text{and} \quad \left[\hat{x}_i, G(\hat{\mathbf{p}})\right] = i\hbar\frac{\partial G}{\partial \hat{p}_i} \quad .$$

Possibly useful:

$$\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\int_{-\infty}^{\infty} du\, u\, e^{-au^2+bu+c} = \frac{d}{db}\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \frac{b}{2a}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\int_{-\infty}^{\infty} du\, u^2\, e^{-au^2+bu+c} = \frac{d^2}{db^2}\int_{-\infty}^{\infty} du\, e^{-au^2+bu+c} = \frac{1}{2a}\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c} + \left(\frac{b}{2a}\right)^2\sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$$

$$\left[\hat{A}, \hat{B}\hat{C}\right] = \left[\hat{A}, \hat{B}\right]\hat{C} + \hat{B}\left[\hat{A}, \hat{C}\right]$$

1. (a) (5) For a harmonic oscillator in a general state with quantum number n, **calculate** $\langle x\rangle$, the expectation value of x.

(b) (5) For this same state $|n\rangle$, **calculate** $\langle p\rangle$.

(c) (15) Now let us focus on a specific state. Suppose that the only thing you know about this state is that

$$\Delta p \Delta x = \frac{\hbar}{2}$$

where

$$(\Delta p)^2 = \langle (p - \langle p\rangle)^2\rangle \quad , \quad (\Delta x)^2 = \langle (x - \langle x\rangle)^2\rangle \ .$$

Determine the minimum energy $\langle H\rangle_{min}$ that this state can have, in terms of $\hbar$ and the angular frequency ω. (Here $H = p^2/2m + m\omega^2 x^2/2$ is the Hamiltonian.)

Hint:

$$\left(\sqrt{A} - \sqrt{B}\right)^2 = A - 2\sqrt{A}\sqrt{B} + B \geq 0 \quad \text{implies that} \quad A + B \geq 2\sqrt{A}\sqrt{B}$$

2. **In this problem, work in "natural" units with** $m = \hbar = \omega = 1$.

(This simplifies the expressions for a and $a^\dagger$ given on the front page, and the expression for the wavefunction.)

A harmonic oscillator is in the state with (unnormalized) wavefunction, in these units,

$$\psi(x) = \left(2x^3 - 3x\right) e^{-x^2/2} \; .$$

(a) (13) **Calculate** the quantum number n of this state, by using the coordinate-representation versions of the operators a and $a^\dagger$ that are given on the front page.

(b) (12) **Calculate** the (unnormalized) wavefunction for the state which is immediately below this state (using the coordinate-representation version of the destruction operator a).

3. (a) (10) Using the uncertainty principle, estimate the ground-state (or zero-point) energy for a particle in a potential .

$$V\,(x) = b|x|^n \; .$$

(Give your answer in terms of the various constants like b and n. Recall that the energy is minimized, with the inequality of the uncertainty principle replaced by an approximate equality for the ground state, and the energy evaluated at characteristic values of momentum and position comparable to the uncertainties.)

(b) (10) What does this reduce to if

$$b = \frac{1}{2} m\omega^2 \quad \text{and} \quad n = 2 \; ?$$

(Your result may emphasize that this approach is really just an estimate.)

4. A particle of mass m and charge q moves in constant crossed magnetic and electric fields, $\mathbf{B} = B_0\hat{\mathbf{z}}$ and $\mathbf{E} = E_0\hat{\mathbf{x}}$. (Here $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the x, y, and z directions.) The Hamiltonian is (in CGS units)

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + q\varphi$$

where $\mathbf{A}$ and φ are the vector and scalar potentials. As usual, $\mathbf{p}$ and $\mathbf{r}$ are vector operators.

(a) (4) Show that the choice of gauge with $\mathbf{A} = B_0 x\hat{\mathbf{y}}$ and $\varphi = -E_0 x$ gives these fields.

(b) (4) Using the Heisenberg equation of motion, determine which of the momenta p_x, p_y, p_z are conserved.

(c) (4) Explain **clearly** why these momenta can be replaced by their eigenvalues.

(So below the notation is changed for these momenta, with p_i being an eigenvalue rather than an operator for each of the conserved momenta.)

(d) (14) Show that the Hamiltonian can be written as the Hamiltonian for a particle which moves with constant velocity in one direction while vibrating as a harmonic oscillator in the plane perpendicular to this direction, with an equilibrium position x_0 that depends on the momentum eigenvalue p_y (and the field strengths B_0 and E_0).

Specifically, you want to obtain a Hamiltonian of the form

$$H = \left[\frac{1}{2m}p_x^2 + \frac{1}{2}m\omega^2\left(x - x_0\right)^2\right] + \left[\frac{1}{2m}p_z^2 + \text{constant}\right]$$

$$\text{constant} = -\frac{mc^2E_0^2}{2B_0^2} - \frac{cp_yE_0}{B_0}$$

where you will determine x_0.

(e) (4) What is the expectation value $\langle\vec{v}\rangle$ of the velocity given by

$$m\vec{v} = \text{mechanical momentum} = \vec{p} - \frac{q}{c}\vec{A} \quad , \quad \vec{p} = \text{canonical momentum}$$

for a state in which the canonical momenta are zero?

You should finally get (after clear arguments)

$$\langle\vec{v}\rangle = -\frac{E_0}{B_0}\,c\,\hat{y}\ .$$

5. (5 extra credit) If $\vec{J} = \vec{L} + \vec{S}$ is the full angular momentum operator, with $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, where the components of $\vec{\sigma}$ are the Pauli matrices (given on the front page), show that $\left[J^2, J_z\right] = 0$.